

Forcing Axioms and the Uniformization-Property.

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Abstract

We show that there are models of MA_{ω_1} where the Σ_3^1 -uniformization property holds. Further we show that “BPFA+ \aleph_1 is not inaccessible to reals” outright implies that the Σ_3^1 -uniformization property is true.

1 Introduction

The question of finding nicely definable choice functions for a definable family of sets is an old and well-studied subject in descriptive set theory. Recall that for an $A \subset 2^\omega \times 2^\omega$, we say that f is a uniformization (or a uniformizing function) of A if there is a function $f : 2^\omega \rightarrow 2^\omega$, $\text{dom}(f) = \text{pr}_1(A)$ and the graph of f is a subset of A .

Definition 1.1. *We say that a pointclass Γ has the uniformization property iff every element of Γ admits a uniformization in Γ .*

It is a classical result due to M. Kondo that lightface Π_1^1 -sets do have the uniformization property, this also yields the uniformization property for Σ_2^1 -sets. This is all ZFC can prove about the uniformization property of projective sets. In the constructible universe L , for every $n \geq 3$, Σ_n^1 does have the uniformization property which follows from the existence of a good wellorder by an old result of Addison (see [2]). Recall that a Δ_n^1 -definable wellorder $<$ of the reals is a *good* Δ_n^1 -wellorder if $<$ is of ordertype ω_1 and the relation $<_I \subset (\omega^\omega)^2$ defined via

$$x <_I y \Leftrightarrow \{(x)_n : n \in \omega\} = \{z : z < y\}$$

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where $(x)_n$ is some fixed recursive partition of x into ω -many reals, is a Δ_n^1 -definable. It is easy to check that the canonical wellorder of the reals in L is a good Δ_2^1 -wellorder so the Σ_n^1 -uniformization property follows.

On the other hand, the axiom of projective determinacy (PD) draws a very different picture. Due to a well-known result of Moschovakis (see [10] 39.9), PD implies that Π_{2n+1}^1 and Σ_{2n+2}^1 -sets have the uniformization property for $n > 1$. The connection of PD with forcing axioms is established via core model induction. Under the assumption of the proper forcing axiom, Schimmerling, Steelt and Woodin showed that PD is true (in fact much more is true, see [12]), thus under PFA the Π_{2n+1}^1 -uniformization holds for $n > 1$. As the uniformization property for one pointclass rules out the uniformization property of the dual pointclass, the behaviour of sets of reals in L and under PFA contradict each other.

The main goal of this paper is to investigate the situation under the weaker forcing axioms MA_{ω_1} and BPFA . We show that for $n = 3$ the situation is orthogonal. Indeed both are compatible with Σ_3^1 -uniformization hence the failure of Π_3^1 -uniformization. If we add the anti-large cardinal assumption $\omega_1 = \omega_1^L$, then the theory BPFA plus “ $\omega_1 = \omega_1^L$ ” implies the Σ_3^1 -uniformization property outright.

Theorem. *Starting with L as our ground model, there is a generic extension of L in which MA_{ω_1} and the Σ_3^1 -uniformization property holds true.*

In the case of BPFA , the additional assumption that ω_1 is not inaccessible to reals in fact yields a strict implication rather than a consistency result.

Theorem. *Assume BPFA and $\omega_1 = \omega_1^{L[r]}$ for some real r . Then every Σ_3^1 set in the plane can be uniformized by a $\Sigma_3^1(r)$ -definable function. In particular, BPFA and $\omega_1 = \omega_1^L$ implies the Σ_3^1 -uniformization property.*

We end the introduction with a short description of how the article is organized. Section 1.1 introduces the forcings we will use, section 1.2 and 1.3 are concerned with producing a suitable ground model which is akin to a similar model used in [8], over which we will start the iteration which shall eventually produce a model for the first main theorem, which is proved in 1.4.

The second section is devoted to the proof of the second main theorem; its proofs are very similar to the according ones in section 1.

1.1 Notation

The notation we use will be mostly standard, we hope. We write $\mathbb{P} = (\mathbb{P}_\alpha : \alpha < \gamma)$ for a forcing iteration of length γ with initial segments \mathbb{P}_α . The α -th

factor of the iteration will be denoted with $\mathbb{P}(\alpha)$. When we feel that there is no danger of confusion we tend to drop the dot on $\mathbb{P}(\alpha)$, even though $\mathbb{P}(\alpha)$ is in fact a \mathbb{P}_α -name of a partial order.

If T is a tree then we often confuse T with the partial order which comes from the tree order. When forcing with this partial order T we will generically add a cofinal branch through the tree T .

We also sometimes write $V[\mathbb{P}] \models \varphi$ to indicate that for every \mathbb{P} -generic filter G over V , $V[G] \models \varphi$.

1.2 The forcings which are used

The forcings which we will use in the construction are all well-known. We nevertheless briefly introduce them and their main properties.

Definition 1.2. *For a stationary $R \subset \omega_1$ the club-shooting forcing for R , denoted by \mathbb{P}_R consists of conditions p which are countable functions from $\alpha + 1 < \omega_1$ to R whose image is a closed set. \mathbb{P}_R is ordered by end-extension.*

The club shooting forcing \mathbb{P}_R is the paradigmatic example for an R -proper forcing, where we say that \mathbb{P} is R -proper if and only if for every condition $p \in \mathbb{P}$, every $\theta > 2^{|\mathbb{P}|}$ (we will utilize the common jargon and say in that situation that θ is sufficiently large) and every countable $M \prec H(\theta)$ such that $M \cap \omega_1 \in R$ and $p, \mathbb{P} \in M$, there is a $q < p$ which is (M, \mathbb{P}) -generic; and a condition $q \in \mathbb{P}$ is said to be (M, \mathbb{P}) -generic if $q \Vdash \dot{G} \cap M$ is an M -generic filter", for \dot{G} the canonical name for the generic filter. See also [6]. The following lemma summarizes several useful properties of the forcing. Its proof is standard so we omit it.

Lemma 1.3. *Let $R \subset \omega_1$ be stationary, co-stationary. Then the club-shooting forcing \mathbb{P}_R generically adds a club through the stationary set $R \subset \omega_1$. Additionally \mathbb{P}_R is R -proper, ω -distributive and hence ω_1 -preserving. Moreover R and all its stationary subsets remain stationary in the generic extension.*

We will choose a family of R_β 's so that we can shoot an arbitrary pattern of clubs through its elements such that this pattern can be read off from the stationarity of the R_β 's in the generic extension. For that it is crucial to recall that for stationary, co-stationary $R \subset \omega_1$, R -proper posets can be iterated with countable support and always yield an R -proper forcing again. This is proved exactly as in the well-known case for plain proper forcings (see [6], Theorem 3.9 and the subsequent discussion).

Fact 1.4. *Let $R \subset \omega_1$ be stationary, co-stationary. Assume that $(\mathbb{P}_\alpha : \alpha < \gamma)$ is a countable support iteration of length γ , let \mathbb{P}_γ denote the resulting partial order and assume also that at every stage α , $\mathbb{P}_\alpha \Vdash \dot{\mathbb{P}}(\alpha)$ is R -proper. Then \mathbb{P}_γ is R -proper.*

Once we decide to shoot a club through a stationary, co-stationary subset of ω_1 , this club will belong to all ω_1 -preserving outer models. This hands us a robust method of coding arbitrary information into a suitably chosen sequence of sets which has been used several times already (see e.g. [5]).

Lemma 1.5. *Let $(R_\alpha : \alpha < \omega_1)$ be a partition of ω_1 into \aleph_1 -many stationary sets, let $r \in 2^{\omega_1}$ be arbitrary, and let \mathbb{P} be a countable support iteration $(\mathbb{P}_\alpha : \alpha < \omega_1)$, inductively defined via*

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{2 \cdot \alpha}} \text{ if } r(\alpha) = 1$$

and

$$\mathbb{P}(\alpha) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{(2 \cdot \alpha) + 1}} \text{ if } r(\alpha) = 0.$$

Then in the resulting generic extension $V[\mathbb{P}]$, we have that $\forall \alpha < \omega_1$:

$$r(\alpha) = 1 \text{ if and only if } R_{2 \cdot \alpha} \text{ is nonstationary,}$$

and

$$r_\alpha = 0 \text{ iff } R_{(2 \cdot \alpha) + 1} \text{ is nonstationary.}$$

Proof. Assume first without loss of generality that $r(0) = 1$, then the first factor of the iteration will be $\mathbb{P}_{\omega_1 \setminus R_0}$ and is $\omega_1 \setminus R_0$ -proper. Now note that $\omega_1 \setminus R_0 \supset R_1$, so $\mathbb{P}_{\omega_1 \setminus R_0}$ is R_1 -proper as well. As the R_α 's form a partition, $\mathbb{P}(\alpha)$ for $\alpha \geq 1$ will always contain R_1 , hence the iteration is R_1 -proper, hence ω_1 preserving.

Now let $\alpha < \omega_1$ be arbitrary and assume that $r(\alpha) = 1$. Then by definition of the iteration we must have shot a club through the complement of $R_{2\alpha}$, thus it is nonstationary in $V[\mathbb{P}]$ as claimed.

On the other hand, if $R_{2\alpha}$ is nonstationary in $V[\mathbb{P}]$, then we assume for a contradiction that we did not use $\mathbb{P}_{\omega_1 \setminus R_{2 \cdot \alpha}}$ in the iteration \mathbb{P} . Note that for $\beta \neq 2 \cdot \alpha$, every forcing of the form $\mathbb{P}_{\omega_1 \setminus R_\beta}$ is $R_{2 \cdot \alpha}$ -proper as $\mathbb{P}_{\omega_1 \setminus R_\beta}$ is $\omega_1 \setminus R_\beta$ -proper and $R_{2 \cdot \alpha} \subset \omega_1 \setminus R_\beta$. Hence the iteration \mathbb{P} will be $R_{2 \cdot \alpha}$ -proper, thus the stationarity of $R_{2 \cdot \alpha}$ is preserved. But this is a contradiction. \square

The second forcing we use is the almost disjoint coding forcing due to R. Jensen and R. Solovay. We will identify subsets of ω with their characteristic function and will use the word reals for elements of 2^ω and subsets of ω

respectively. Let $D = \{d_\alpha \mid \alpha < \aleph_1\}$ be a family of almost disjoint subsets of ω , i.e. a family such that if $r, s \in D$ then $r \cap s$ is finite. Let $X \subset \kappa$ for $\kappa \leq 2^{\aleph_0}$ be a set of ordinals. Then there is a ccc forcing, the almost disjoint coding $\mathbb{A}_D(X)$ which adds a new real x which codes X relative to the family D in the following way

$$\alpha \in X \text{ if and only if } x \cap d_\alpha \text{ is finite.}$$

Definition 1.6. *The almost disjoint coding $\mathbb{A}_D(X)$ relative to an almost disjoint family D consists of conditions $(r, R) \in \omega^{<\omega} \times D^{<\omega}$ and $(s, S) < (r, R)$ holds if and only if*

1. $r \subset s$ and $R \subset S$.
2. If $\alpha \in X$ and $d_\alpha \in R$ then $r \cap d_\alpha = s \cap d_\alpha$.

For the rest of this paper we let $D \in L$ be the definable almost disjoint family of reals one obtains when recursively adding the $<_L$ -least real to the family which is almost disjoint from all the previously picked reals. Whenever we use almost disjoint coding forcing, we assume that we code relative to this fixed almost disjoint family D .

The last two forcings we briefly discuss are Jech's forcing for adding a Suslin tree with countable conditions and, given a Suslin tree T , the associated forcing which adds a cofinal branch through T . Recall that a set theoretic tree $(T, <)$ is a Suslin tree if it is a normal tree of height ω_1 and has no uncountable antichain. As a result, forcing with a Suslin tree S , where conditions are just nodes in S , and which we always denote with S again, is a ccc forcing of size \aleph_1 . Jech's forcing to generically add a Suslin tree is defined as follows.

Definition 1.7. *Let \mathbb{P}_J be the forcing whose conditions are countable, normal trees ordered by end-extension, i.e. $T_1 < T_2$ if and only if $\exists \alpha < \text{height}(T_1) T_2 = \{t \upharpoonright \alpha : t \in T_1\}$*

It is wellknown that \mathbb{P}_J is σ -closed and adds a Suslin tree. In fact more is true, the generically added tree T has the additional property that for any Suslin tree S in the ground model $S \times T$ will be a Suslin tree in $V[G]$. This can be used to obtain a robust coding method (see also [7] for more applications)

Lemma 1.8. *Let V be a universe and let $S \in V$ be a Suslin tree. If \mathbb{P}_J is Jech's forcing for adding a Suslin tree and if T is the generic tree then*

$$V[\mathbb{P}_J][T] \models S \text{ is Suslin.}$$

Proof. Let \dot{T} be the \mathbb{P}_J -name for the generic Suslin tree. We claim that $\mathbb{P}_J * \dot{T}$ has a dense subset which is σ -closed. As σ -closed forcings will always preserve ground model Suslin trees, this is sufficient. To see why the claim is true consider the following set:

$$\{(p, \check{q}) : p \in \mathbb{P}_J \wedge \text{height}(p) = \alpha + 1 \wedge \check{q} \text{ is a node of } p \text{ of level } \alpha\}.$$

It is easy to check that this set is dense and σ -closed in $\mathbb{P}_J * \dot{T}$. □

A similar observation shows that we can add an ω_1 -sequence of such Suslin trees with a countably supported iteration.

Lemma 1.9. *Let S be a Suslin tree in V and let \mathbb{P} be a countably supported product of length ω_1 of forcings \mathbb{P}_J with G its generic filter. Then in $V[G]$ there is an ω_1 -sequence of Suslin trees $\vec{T} = (T_\alpha : \alpha \in \omega_1)$ such that for any finite $e \subset \omega$ the tree $S \times \prod_{i \in e} T_i$ will be a Suslin tree in $V[G]$.*

These sequences of Suslin trees will be used for coding in our proof and get a name.

Definition 1.10. *Let $\vec{T} = (T_\alpha : \alpha < \kappa)$ be a sequence of Suslin trees. We say that the sequence is an independent family of Suslin trees if for every finite set $e = \{e_0, e_1, \dots, e_n\} \subset \kappa$ the product $T_{e_0} \times T_{e_1} \times \dots \times T_{e_n}$ is a Suslin tree again, provided the e_i 's are different.*

1.3 The ground model W

We have to first create a suitable ground model W over which the actual iteration will take place. The construction is inspired by [8], where a very similar universe is used as well. W will be a generic extension of L , satisfying CH and has the crucial property that in W there is an ω_2 -sequence \vec{S} of independent Suslin trees which is $\Sigma_1(\omega_1)$ -definable over $H(\omega_2)^W$. The sequence \vec{S} will enable a coding method which is independent of the surrounding universe, a feature we will exploit to a great extent in the upcoming.

To form W , we start with Gödel's constructible universe L as our ground model. Recall that L comes equipped with a Σ_1 -definable, global well-order $<_L$ of its elements. We first fix an appropriate sequence of stationary, co-stationary subsets of ω_1 using Jensen's \diamond -sequence. The proof of the next lemma is well-known so we skip it.

Fact 1.11. *In L there is a sequence $(a_\alpha : \alpha < \omega_1)$ of countable subsets of ω_1 such that any set $A \subset \omega_1$ is guessed stationarily often by the a_α 's, i.e. $\{\alpha < \omega_1 : a_\alpha = A \cap \alpha\}$ is a stationary and co-stationary subset of ω_1 . The sequence $(a_\alpha : \alpha < \omega_1)$ can be defined in a Σ_1 way over the structure L_{ω_1} .*

The \diamond -sequence can be used to produce an easily definable sequence of L -stationary, co-stationary subsets of ω_1 : we list the elements of L_{ω_2} in an ω_2 sequence $(r_\alpha : \alpha < \omega_2)$. We fix

$$R := \{\alpha < \omega_1 : a_\alpha = r_0 \cap \alpha\}$$

and note that R is stationary and co-stationary.

Then we define for every $\beta < \omega_2$, $\beta \neq 0$ a stationary, co-stationary set in the following way:

$$R'_\beta := \{\alpha < \omega_1 : a_\alpha = r_\beta \cap \alpha\}$$

and

$$R_\beta := \{\alpha < \omega_1 : a_\alpha = r_\beta \cap \alpha\} \setminus R.$$

It is clear that $\forall \alpha \neq \beta (R_\alpha \cap R_\beta \in \text{NS}_{\omega_1})$ and that $R_\beta \cap R = \emptyset$. To avoid writing $\beta \neq 0$ all the time we re-index and confuse $(R_\beta : \beta < \omega_2, \beta \neq 0)$ with $(R_\beta : \beta < \omega_2)$. We derive the following standard result concerning the definability of the R_β 's:

Lemma 1.12. *For any $\beta < \omega_2$, membership in R_β is uniformly $\Sigma_1(\omega_1)$ -definable over the model L_{ω_2} , i.e. there is a Σ_1 -formula with ω_1 as a parameter $\psi(v_0, v_1, \omega_1)$ such that for every $\beta < \omega_2$, $(\alpha \in R_\beta \Leftrightarrow L_{\omega_2} \models \psi(\alpha, \beta, \omega_1))$.*

Proof. First we note that there is a $\Sigma_1(\omega_1)$ -formula $\theta'(\eta, x, \omega_1)$ for which $L_{\omega_2} \models \theta'(\eta, x, \omega_1)$ is true if and only if “ x is the η -th subset of ω_1 in $<_L$, the canonical L -wellorder”. It follows that there is a $\Sigma_1(\omega_1)$ -formula $\theta(\eta, \zeta, x, \omega_1)$ for which $L_{\omega_2} \models \theta(\eta, \zeta, x, \omega_1)$ is true if and only if “ x equals $r_\eta \cap \zeta$ ”. Further recall that using the last fact we found already a Σ_1 -formula, let us denote it with $\varphi(\xi, y)$, such that $L_{\omega_1} \models \varphi(\xi, y)$ holds if and only if “ y is the ξ -th element of the canonical \diamond -sequence”.

Then membership in R'_β can be expressed using the following formula:

$$\alpha \in R'_\beta \Leftrightarrow L_{\omega_2} \models \exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x, \omega_1))$$

Note here that actually every \aleph_1 -sized L_γ , for γ a limit ordinal, which models (a sufficiently big fragment of) ZF^- and contains α and β is sufficient to witness membership of α in R'_β using the formula $\exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x, \omega_1))$. In particular, not being an element of R can be written in a $\Sigma_1(\omega_1)$ fashion as well:

$$\alpha \notin R \Leftrightarrow L_{\omega_2} \models \exists \gamma > \omega_1 (L_\gamma \models \text{ZF}^- \wedge \exists x(\varphi(\alpha, x) \wedge \neg \theta(0, \alpha, x, \omega_1)))$$

It follows that membership in R_β allows this representation:

$$\begin{aligned} \alpha \in R_\beta \Leftrightarrow L_{\omega_2} \models & \exists x(\varphi(\alpha, x) \wedge \theta(\beta, \alpha, x, \omega_1)) \wedge \\ & \exists \gamma > \omega_1 (L_\gamma \models \text{“ZF}^- \wedge \\ & \exists y(\varphi(\eta, y) \wedge \neg\theta(0, \alpha, y, \omega_1)\text{”}) \end{aligned}$$

Note that the last formula is $\Sigma_1(\omega_1)$, thus we found our desired $\psi(v_0, v_1, \omega_1)$. \square

We proceed with defining the universe W . First we generically add \aleph_2 -many Suslin trees using of Jech’s Forcing \mathbb{P}_J . We let

$$\mathbb{Q}^0 := \prod_{\beta \in \omega_2} \vec{\mathbb{P}}_J$$

using countable support. This is a σ -closed, hence proper notion of forcing. We denote the generic filter of \mathbb{Q}^0 with $\vec{S} = (S_\alpha : \alpha < \omega_2)$ and note that by Lemma 1.9 \vec{S} is independent. We fix a definable bijection between $[\omega_1]^\omega$ and ω_1 and identify the trees in $(S_\alpha : \alpha < \omega_1)$ with their images under this bijection, so the trees will always be subsets of ω_1 from now on.

In a second step we code the trees from \vec{S} into the sequence of L -stationary subsets \vec{R} we produced earlier, using the method introduced in Lemma 1.5. It is important to note, that the forcing we are about to define does preserve Suslin trees, a fact we will show later. The forcing used in the second step will be denoted by \mathbb{Q}^1 and will itself be a countable support iteration of length ω_1 whose factors are itself countably supported iterations of length ω_1 . Fix first a definable bijection $h \in L_{\omega_3}$ between $\omega_1 \times \omega_2$ and ω_2 and write \vec{R} from now on in ordertype $\omega_1 \cdot \omega_2$ making implicit use of h , so we assume that $\vec{R} = (R_\alpha : \alpha < \omega_1 \cdot \omega_2)$.

\mathbb{Q}^1 is defined via induction in the ground model $L[\mathbb{Q}^0]$. Assume we are at stage $\alpha < \omega_2$ and we have already created the iteration \mathbb{Q}_α^1 up to stage α . We work with $L[\mathbb{Q}^0][\mathbb{Q}_\alpha^1]$ as our ground model and consider the Suslin tree $S_\alpha \subset \omega_1$. We define the forcing we want to use at stage α , denoted by $\mathbb{Q}^1(\alpha)$, as the countable support iteration which codes the characteristic function of S_α into the α -th ω_1 -block of the R_β ’s just as in Lemma 1.5. So $\mathbb{Q}^1(\alpha) = \star_{\gamma < \omega_1} \mathbb{R}_\alpha(\gamma)$ is again a countable support iteration in $L[\mathbb{Q}^0][\mathbb{Q}_\alpha^1]$, defined inductively via

$$\forall \gamma < \omega_1 (\mathbb{R}_\alpha(\gamma) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{\omega_1 \cdot \alpha + 2\gamma + 1}}) \text{ if } S_\alpha(\gamma) = 0$$

and

$$\forall \gamma < \omega_1 (\mathbb{R}_\alpha(\gamma) := \dot{\mathbb{P}}_{\omega_1 \setminus R_{\omega_1 \cdot \alpha + 2\gamma}}) \text{ if } S_\alpha(\gamma) = 1.$$

Recall that we let R be a stationary, co-stationary subset of ω_1 which is disjoint from all the R_α 's which are used. We let \mathbb{Q}^1 be the countably supported iteration,

$$\mathbb{Q}^1 := \star_{\alpha < \omega_2} \mathbb{Q}_\alpha^1$$

which is an R -proper forcing. We shall see later that \mathbb{Q}^1 in fact is ω -distributive, hence the iteration \mathbb{Q}^1 is in fact a countably supported product. This way we can turn the generically added sequence of Suslin trees \vec{S} into a definable sequence of Suslin trees. Indeed, if we work in $L[\vec{S} * G]$, where $\vec{S} * G$ is $\mathbb{Q}^0 * \mathbb{Q}^1$ -generic over L , then, as seen in Lemma 1.5

$$\begin{aligned} \forall \alpha < \omega_2, \gamma < \omega_1 (\gamma \in S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma} \text{ is not stationary and} \\ \gamma \notin S_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \gamma + 1} \text{ is not stationary}) \end{aligned}$$

Note here that the above formula can be used to make every $S_\alpha \Sigma_1(\omega_1, \alpha)$ definable over $L[\vec{S} * G]$ as is shown with the next lemma.

Lemma 1.13. *There is a $\Sigma_1(\omega_1)$ -formula $\Phi(v_0, v_1, \omega_1)$ such that every $\gamma < \omega_2$,*

$$L[\vec{S} * G] \models \forall \beta < \omega_1 (\Phi(\beta, \gamma, \omega_1) \Leftrightarrow \beta \in S_\gamma)$$

*Thus initial segments of the sequence \vec{S} are uniformly $\Sigma_1(\omega_1)$ -definable over $L[\vec{S} * G]$.*

Proof. Let $\gamma < \omega_2$. We claim that already \aleph_1 -sized, transitive models of \mathbf{ZF}^- which contain a club through the complement of exactly one element of every pair $\{(R_\alpha, R_{\alpha+1}) : \alpha < \omega_1 \cdot \gamma\}$ are sufficient to compute correctly $\vec{S} \upharpoonright \gamma$ via the following $\Sigma_1(\gamma, \omega_1)$ -formula:

$$\begin{aligned} \Psi(X, \gamma, \omega_1) \equiv \exists M (M \text{ transitive} \wedge M \models \mathbf{ZF}^- \wedge \omega_1, \gamma \in M \wedge \\ M \models \forall \beta < \omega_1 \cdot \gamma (\text{either } R_{2\beta} \text{ or } R_{2\beta+1} \text{ is nonstationary}) \wedge \\ M \models X \text{ is an } \omega_1 \cdot \gamma\text{-sequence } (X_\alpha)_{\alpha < \omega_1 \cdot \gamma} \text{ of subsets of } \omega_1 \wedge \\ M \models \forall \alpha, \delta (\delta \in X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \delta} \text{ is not stationary and} \\ \delta \notin X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \delta + 1} \text{ is not stationary}) \end{aligned}$$

We want to show that

$$X = \vec{S} \upharpoonright \gamma \text{ if and only if } \Psi(X, \gamma, \omega_1) \text{ is true in } L[\vec{S} * G].$$

For the backwards direction, we assume that M is a model and $X \in M$ is a set, as on the right hand side of the above. We shall show that indeed

$X = \vec{S} \upharpoonright \gamma$. As M is transitive and a model of \mathbf{ZF}^- it will compute every R_β , $\beta < \omega_1 \cdot \gamma$ correctly by Lemma 1.12. As being nonstationary is a $\Sigma_1(\omega_1)$ -statement, and hence upwards absolute, we conclude that if M believes to see a pattern written into (its versions of) the R_β 's, this pattern is exactly the same as is seen by the real world $L[\vec{S} * G]$. But we know already that in $L[\vec{S} * G]$, the sequence \vec{S} is written into the R_β 's, thus $X = \vec{S} \upharpoonright \gamma$ follows.

On the other hand, if $X = \vec{S} \upharpoonright \gamma$, then

$$\begin{aligned} L[\vec{S} * G] &\models \forall \beta < \omega_1 \cdot \gamma (\text{either } R_{2\beta} \text{ or } R_{2\beta+1} \text{ is nonstationary}) \\ L[\vec{S} * G] &\models X \text{ is an } \omega_1 \cdot \gamma\text{-sequence } (X_\alpha)_{\alpha < \omega_1 \cdot \gamma} \text{ of subsets of } \omega_1 \end{aligned}$$

and

$$\begin{aligned} L[\vec{S} * G] &\models \forall \alpha, \delta < \omega_1 (\delta \in X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \delta} \text{ is not stationary and} \\ &\quad \delta \notin X_\alpha \Leftrightarrow R_{\omega_1 \cdot \alpha + 2 \cdot \delta + 1} \text{ is not stationary}) \end{aligned}$$

By reflection, there is an \aleph_1 -sized, transitive model M which models the assertions above, which gives the direction from left to right. \square

Let us set

$$W := L[\mathbb{Q}^0 * \mathbb{Q}^1]$$

which will serve as our ground model for a second iteration of length ω_2 . We shall need the following well-known result (see [1], Theorem 2.10 pp.20, or see [5] Lemma 12).

Fact 1.14. *Let $R \subset \omega_1$ be stationary and co-stationary. Assume CH and let $(\mathbb{P}_\alpha : \alpha \leq \delta)$ be a countable support iteration of R -proper posets such that for every $\alpha \leq \delta$*

$$\mathbb{P}_\alpha \Vdash |\mathbb{P}(\alpha)| = \aleph_1.$$

Then \mathbb{P}_δ satisfies the \aleph_2 -cc.

Lemma 1.15. *W is an ω -distributive generic extension of L which also satisfies $\aleph_2^L = \aleph_2^W$.*

Proof. The second assertion follows immediately from the last Fact. The first assertion holds by the following argument which already will look familiar. First note that as \mathbb{Q}^0 does not add any reals it is sufficient to show that \mathbb{Q}^1 is ω -distributive. Let $p \in \mathbb{Q}^1$ be a condition and assume that $p \Vdash \dot{r} \in 2^\omega$. We shall find a stronger $q < p$ and a real r in the ground model such that $q \Vdash \check{r} = \dot{r}$. Let $M \prec H(\omega_3)$ be a countable elementary submodel which

contains p, \mathbb{Q}^1 and \dot{r} and such that $M \cap \omega_1 \in R$, where R is our fixed stationary set from above. Inside M we recursively construct a decreasing sequence p_n of conditions in \mathbb{Q}^1 , such that for every n in ω , $p_n \in M$, p_n decides $\dot{r}(n)$ and for every α in the support of p_n , the sequence $\sup_{n \in \omega} \max(p_n(\alpha))$ converges towards $M \cap \omega_1$ which is in R . Now, $q' := \bigcup_{n \in \omega} p_n$ and for every $\alpha < \omega_1$ such that $q'(\alpha) \neq 1$ (where 1 is the weakest condition of the forcing), in other words for every α in the support of q' we define $q(\alpha) := q'(\alpha) \cup \{(M \cap \omega_1), (M \cap \omega_1)\}$ and $q(\alpha) = 1$ otherwise. Then $q = (q(\alpha))_{\alpha < \omega_1}$ is a condition in \mathbb{Q}^1 , as can be readily verified and $q \Vdash \dot{r} = \check{r}$, as desired.

The second assertion follows immediately from the well-known theorem that under CH, a countable support iteration of R -proper forcings of size \aleph_1 , \square

Note that by the last lemma, the second forcing \mathbb{Q}^1 which is an the countably supported iteration of the appropriate club shooting forcings is in fact just a countably supported product of its factors, i.e. at every stage of the iteration we can force with the according \mathbb{P}_R , as computed in L .

Our goal is to use \vec{S} for coding again. For this it is essential, that the sequence remains independent in W . To see this we shall argue that forcing with \mathbb{Q}^1 over $L[\mathbb{Q}^0]$ preserves Suslin trees. The following line of reasoning is similar to [7]. Recall that for a forcing \mathbb{P} , θ sufficiently large and regular and $M \prec H(\theta)$, a condition $q \in \mathbb{P}$ is (M, \mathbb{P}) -generic iff for every maximal antichain $A \subset \mathbb{P}$, $A \cap M$ is predense below q . In the following we will write T_η to denote the η -th level of the tree T and $T \upharpoonright \eta$ to denote the set of nodes of T of height $< \eta$. The key fact is the following (see [11] for the case where \mathbb{P} is proper)

Lemma 1.16. *Let T be a Suslin tree, $R \subset \omega_1$ stationary and \mathbb{P} an R -proper poset. Let θ be a sufficiently large cardinal. Then the following are equivalent:*

1. $\Vdash_{\mathbb{P}} T$ is Suslin
2. if $M \prec H_\theta$ is countable, $\eta = M \cap \omega_1 \in R$, and \mathbb{P} and T are in M , further if $p \in \mathbb{P} \cap M$, then there is a condition $q < p$ such that for every condition $t \in T_\eta$, (q, t) is $(M, \mathbb{P} \times T)$ -generic.

Proof. For the direction from left to right note first that $\Vdash_{\mathbb{P}} T$ is Suslin implies $\Vdash_{\mathbb{P}} T$ is ccc, and in particular it is true that for any countable elementary submodel $N[\dot{G}_{\mathbb{P}}] \prec H(\theta)^{V[\dot{G}_{\mathbb{P}}]}$, $\Vdash_{\mathbb{P}} \forall t \in T (t \text{ is } (N[\dot{G}_{\mathbb{P}}], T)\text{-generic})$. Now if $M \prec H(\theta)$ and $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, T \in M$ and $p \in \mathbb{P} \cap M$ then there is a $q < p$ such that q is (M, \mathbb{P}) -generic. So $q \Vdash \forall t \in T (t \text{ is } (M[\dot{G}_{\mathbb{P}}], T)\text{-generic})$, and this in particular implies that (q, t) is $(M, \mathbb{P} \times T)$ -generic for all $t \in T_\eta$.

For the direction from right to left assume that $\Vdash \dot{A} \subset T$ is a maximal antichain. Let $B = \{(x, s) \in \mathbb{P} \times T : x \Vdash_{\mathbb{P}} \dot{s} \in \dot{A}\}$, then B is a predense subset in $\mathbb{P} \times T$. Let θ be a sufficiently large regular cardinal and let $M \prec H(\theta)$ be countable such that $M \cap \omega_1 = \eta \in R$ and $\mathbb{P}, B, p, T \in M$. By our assumption there is a $q <_{\mathbb{P}} p$ such that $\forall t \in T_\eta((q, t)$ is $(M, \mathbb{P} \times T)$ -generic). So $B \cap M$ is predense below (q, t) for every $t \in T_\eta$, which yields that $q \Vdash_{\mathbb{P}} \forall t \in T_\eta \exists s <_T t (s \in \dot{A})$ and hence $q \Vdash \dot{A} \subset T \upharpoonright \eta$, so $\Vdash_{\mathbb{P}} T$ is Suslin. \square

We strongly believe that in a similar way, one can show that Theorem 1.3 of [11] holds true if we replace proper by R -proper for $R \subset \omega_1$ a stationary subset, i.e. that a countable support iteration of R -proper forcings which preserve Suslin trees results in a forcing which preserves Suslin trees. Instead of proving this we will take a more direct route though and prove that \mathbb{Q}^1 preserves Suslin trees by hand.

Lemma 1.17. *Let $R \subset \omega_1$ be stationary, co-stationary, then the club shooting forcing \mathbb{P}_R preserves Suslin trees.*

Proof. Let T be an arbitrary Suslin tree from the ground model V . Because of Lemma 1.16, it is enough to show that for any regular and sufficiently large θ , every $M \prec H_\theta$ with $M \cap \omega_1 = \eta \in R$, and every $p \in \mathbb{P}_R \cap M$ there is a $q < p$ such that for every $t \in T_\eta$, (q, t) is $(M, (\mathbb{P}_R \times T))$ -generic. Note first that, as T is Suslin, every node $t \in T_\eta$ is an (M, T) -generic condition. Further, as forcing with a Suslin tree is ω -distributive, $(H(\omega_1))^{M[G]} = (H(\omega_1))^M$ for every T -generic filter G over V . As $\mathbb{P}_R \subset H(\omega_1)$, we obtain that the set $M[G] \cap \mathbb{P}_R$ is independent of the choice of the generic filter G and equals $M \cap \mathbb{P}_R$. Likewise $M[G] \cap \omega_1 = M \cap \omega_1$, for every T -generic filter.

Next we note that for a countable M and a V -generic filter $G \subset T$, the model $M[G]$ is (up to isomorphism) uniquely determined by the $t \in T_\eta$, such that $t \in G$ and $\eta = M \cap \omega_1$. This is clear as we can transitively collapse $M[G]$ to obtain a structure of the form $\bar{M}[t]$, where \bar{M} is the image of M under the collapse map and $t \in T_\eta$ is the unique node in T to which G is sent to by the collapse map. So for a countable $M \prec H(\theta)$, and $\eta = M \cap \omega_1$, we write $M[t]$ for the unique model of the form $M[G]$, for G T -generic over V and $t \in G \cap T_\eta$. With an argument almost identical to the one used in the proof of Lemma 2.2 it is not hard to see that if $M \prec H(\theta)$ is such that $M \cap \omega_1 \in R$ then an ω -length descending sequence of \mathbb{P}_R -conditions in M whose domains converge to $M \cap \omega_1$ has a lower bound as $M \cap \omega_1 \in R$.

We construct an ω -sequence of elements of \mathbb{P}_R which has a lower bound which will be the desired condition q such that for every $t \in T_\eta$, (q, t) is $(M, \mathbb{P}_R \times T)$ -generic. We list the nodes on T_η , $(t_i : i \in \omega)$ and consider

the according generic extensions $M[t_i]$. In every $M[t_i]$ we list the \mathbb{P}_R -dense subsets of $M[t_i]$, $(D_n^{t_i} : n \in \omega)$, write the so listed dense subsets of $M[t_i]$ as an $\omega \times \omega$ -matrix and enumerate this matrix in an ω -length sequence of dense sets $(D_i : i \in \omega)$. If $p = p_0 \in \mathbb{P}_R \cap M$ is arbitrary we can find, using the fact that $\forall i (\mathbb{P}_R \cap M[t_i] = M \cap \mathbb{P}_R)$, an ω -length, descending sequence of conditions below p_0 in $\mathbb{P}_R \cap M$, $(p_i : i \in \omega)$ such that $p_{i+1} \in M \cap \mathbb{P}_R$ is in D_i . By the usual density argument we can conclude that the domain of the conditions p_i converge to $M[t_i] \cap \omega_1 = M \cap \omega_1$. Then the p_i 's have a lower bound $q = p_\omega \in \mathbb{P}_R$, namely $q = \bigcup_{i \in \omega} p_i \cup \{(\eta, \eta)\}$ and (t, q) is an $(M, T \times \mathbb{P}_R)$ -generic condition for every $t \in T_\eta$ as any $t \in T_\eta$ is (M, T) -generic and every such t forces that q is $(M[T], \mathbb{P}_R)$ -generic; moreover $q < p$ as desired. \square

The above quickly generalizes to \mathbb{Q}^1 .

Lemma 1.18. *The second forcing \mathbb{Q}^1 we use to produce $W = L[\mathbb{Q}^0 * \mathbb{Q}^1]$ preserves Suslin trees.*

Proof. We work over the ground model $L[\mathbb{Q}^0]$. Let T be a Suslin tree from $L[\mathbb{Q}^0]$. As by definition of \mathbb{Q}^1 , there is an $R \subset \omega_1$ for which \mathbb{Q}^1 is R -proper, we, again, shall show that for any regular and sufficiently large θ , every $M \prec H_\theta$ with $M \cap \omega_1 = \eta \in R$, and every $p \in \mathbb{Q}^1 \cap M$ there is a $q < p$ such that for every $t \in T_\eta$, (q, t) is $(M, (\mathbb{Q}^1 \times T))$ -generic.

We note that by the remark following the proof of Lemma 1.17, \mathbb{Q}^1 is equivalent to the countably supported product of its factors. As a consequence of this and the fact that forcing with a Suslin tree is ω -distributive, we obtain that for any $M \prec H(\theta)$, and any G which is T -generic over $L[\mathbb{Q}^0]$, $H(\omega_1)^{M[G]} = H(\omega_1)^M$ and as $\mathbb{Q}^1 \subset H(\omega_1)^{L[\mathbb{Q}^0]}$, it must hold that $\mathbb{Q}^1 \cap M[G] = \mathbb{Q}^1 \cap M$, and is thus independent of G .

The proof now follows closely the one of the last Lemma, and uses its notions. We start with an arbitrary $p \in \mathbb{Q}^1 \cap M$ and need to find a $q < p$ such that (q, t) is $(M, \mathbb{Q}^1 \times T)$ -generic for every $t \in T_\eta$. We first list all T_η -nodes $(t_i)_{i \in \omega}$. In every $M[t_i]$, $t \in T_\eta$, we list the dense sets $\{D_n^{t_i} \subset \mathbb{Q}^1 : n \in \omega\}$ and re-list the resulting $\omega \times \omega$ matrix of dense subsets in ordertype ω . Now, starting with our condition $p := p_0$ we form a descending sequence of conditions such that for every $n \in \omega$, $p_n \in D_n$. By density the support of p_n will converge to $M \cap \omega_1 = \eta$ as n grows, and likewise the maximum of the domain and the maximum of the range of every coordinate of p_n will converge to η as well. Thus $q := \bigcup_{n \in \omega} p_n \cup \{(\eta, \eta, \xi) : \xi < \eta\}$ is a condition in \mathbb{Q}^1 below p . By construction, every $t \in T_\eta$ forces that q is $(M[t], \mathbb{Q}^1)$ -generic, and as forcing with T has the ccc every $t \in T_\eta$ is trivially (M, T) -generic. As a consequence, (q, t) is $(M, \mathbb{Q}^1 \times T)$ -generic for every $t \in T_\eta$ as desired. \square

To summarize the main results from above

Theorem 1.19. *The universe $W = L[\mathbb{Q}^0][\mathbb{Q}^1]$ is an ω -distributive, \aleph_2 -preserving generic extension of L and contains \vec{S} which is an independent sequence of Suslin trees. Moreover, for arbitrary $\alpha < \omega_2$, $\vec{S} \upharpoonright \alpha$ is uniformly $\Sigma_1(\alpha, \omega_1)$ -definable over W .*

We end with a straightforward lemma which is used later in coding arguments.

Lemma 1.20. *Let T be a Suslin tree and let $\mathbb{A}_D(X)$ be the almost disjoint coding which codes a subset X of ω_1 into a real with the help of an almost disjoint family of reals D of size \aleph_1 . Then*

$$\mathbb{A}_D(X) \Vdash T \text{ is Suslin}$$

holds.

Proof. This is clear as $\mathbb{A}_D(X)$ has the Knaster property, thus the product $\mathbb{A}_D(X) \times T$ is ccc and T must be Suslin in $V[\mathbb{A}_D(X)]$. \square

1.4 Coding over W to produce \tilde{W}

We shall use the just created W to start a second iteration which will code a fresh sequence of Suslin trees \vec{T} into the W -definable sequence \vec{S} of Suslin trees. The reason for this seemingly redundant choice, is that we later want to force \mathbf{MA}_{ω_1} , which of course will necessarily create a lot of noise on any definable sequence of independent Suslin trees. This noise is a threat to any intentional coding we aim to build in order to have a universe for the Σ_3^1 -uniformization property.

The addition of the second sequence \vec{T} of independent Suslin trees enables us to later use coding with \vec{T} trees along the usual forcing which produces \mathbf{MA}_{ω_1} and yet be in full control of the codes we create on the \vec{T} -sequence. This effect can not be reproduced when forcing over just W , this is why we need to pass from W to a more suitable generic extension.

The universe $\tilde{W} = W[\mathbb{P}_0 * \mathbb{P}_1]$ is a two step generic extension of W . The first factor \mathbb{P}_0 is just an ω_2 -length product of ordinary Cohen forcing with finite support

$$\mathbb{P}_0 := \prod_{\alpha < \omega_2} \mathbb{C}.$$

As is known due to S. Shelah (see [9], Theorem 28.12, Lemma 28.13) every generically added Cohen real r can be used to define a new Suslin tree T_r . To be more precise, if V is our ground model and

$$(e_\alpha : \alpha < \omega_1)$$

is an ω_1 sequence of functions in V which satisfies

1. for every $\alpha < \omega_1$, e_α is an injection from α to ω ;
2. and for every $\alpha < \beta < \omega_1$, $e_\alpha(\xi) = e_\beta(\xi)$ for all but finitely many $\xi < \alpha$,

then

$$T_r := \{r \circ (e_\alpha \upharpoonright \beta) : \alpha, \beta < \omega_1\}$$

is a Suslin tree, as long as r is a Cohen real over V .

Consequentially, if $(c_\alpha : \alpha < \omega_2)$ denotes the sequence of Cohen reals in $W[\mathbb{P}_0]$, the sequence of trees

$$\vec{T} := (T_\alpha : \alpha < \omega_2)$$

is an independent sequence of Suslin trees. Indeed, any finite product of T_α 's must be a Suslin tree again, as otherwise, if say T_1 and T_2 are such that $T_1 \times T_2$ is not Suslin, then $\mathbb{C} \times \mathbb{C} \times T_1 \times T_2$ is not a ccc forcing. But rearranging the factors yields that T_2 is a Suslin tree in $V[\mathbb{C}][T_1][\mathbb{C}]$, hence $\mathbb{C} \times T_1 \times \mathbb{C} \times T_2$ has the ccc, which is a contradiction.

Our second forcing \mathbb{P}_1 uses our already created independent sequence \vec{S} to code up the trees from \vec{T} with a method completely analogous to our forcing \mathbb{Q}^1 , we used in the definition of W . That is if we fix an arbitrary $\alpha < \omega_2$ with the according tree $T_\alpha \subset \omega_1$, and for $\beta < \omega_1$, we let

$$\mathbb{P}_{1,\alpha}^\beta = \begin{cases} S_{\omega_1\alpha+\beta} & \text{if } T_\alpha(\beta) = 1 \\ \text{SP}(S_{\omega_1\alpha+\beta}) & \text{if } T_\alpha(\beta) = 0 \end{cases}$$

where $S_{\omega_1\alpha+\beta}$ just denotes the forcing which adds an ω_1 -branch through $S_{\omega_1\alpha+\beta}$, and $\text{SP}(S_{\omega_1\alpha+\beta})$ is the forcing which specializes $S_{\omega_1\alpha+\beta}$.

Then

$$\mathbb{P}_{1,\alpha} = \prod_{\beta < \omega_1} \mathbb{P}_{1,\alpha}^\beta$$

using finite support. Note that for every $\alpha < \omega_2$, $\mathbb{P}_{1,\alpha}$ has the ccc.

Finally we let

$$\mathbb{P}^1 := \prod_{\alpha < \omega_2} \mathbb{P}_{1,\alpha}$$

again using finite support. The upshot of these manipulations is, that \vec{T} is now a uniformly definable ω_2 -sequence of independent Suslin trees which has a second crucial feature, we discuss later and which is not shared by \vec{S} .

We let $\tilde{W} = W[\mathbb{P}_0][\mathbb{P}_1]$.

Lemma 1.21. *The universe \tilde{W} is a cardinal preserving generic extension of L , whose continuum is \aleph_2 . Moreover there is a $\Sigma_1(\omega_1)$ -formula $\Xi(v_0, v_1, \omega_1)$ such that every $\gamma < \omega_2$,*

$$\tilde{W} \models \forall \beta < \omega_1 (\Xi(\beta, \gamma, \omega_1) \Leftrightarrow \beta \in T_\gamma)$$

Thus initial segments of the sequence \vec{T} are uniformly $\Sigma_1(\omega_1)$ -definable over \tilde{W} .

Proof. The proof of the definability claim is almost identical to the proof of Lemma 1.13, so we omit it. The rest is just a summary of the discussions above. \square

1.5 Iteration over \tilde{W}

Having defined \tilde{W} we start an iteration of length ω_2 . In the resulting universe MA_{ω_1} and the Σ_3^1 -uniformization property will be true.

We isolate first a class of transitive models which we take advantage of when showing the Σ_3^1 -uniformization property.

The following is inspired by a similar definition in [4]: Let \mathbb{T} be the theory consisting of the following sentences:

- ZF^- ,
- $\forall x (|x| \leq \aleph_1)$,
- Every $X \subset \omega_1$ is coded into a real r_X with the help of the canonical almost disjoint family D of reals from L .
- For every real r there is an ordinal γ such that the block $(T_{\gamma \cdot \omega + n} : n \in \omega)$ codes r via

$$\begin{aligned} \forall n \in \omega (n \in r \Leftrightarrow T_{\omega \gamma + n} \text{ has a cofinal branch} \wedge \\ n \notin r \Leftrightarrow T_{\omega \gamma + n} \text{ is special}) \end{aligned}$$

- For every pair of ordinals $(2\beta, 2\beta + 1)$ either $R_{2\beta}$ or $R_{2\beta+1}$, where S_β is our definable L -stationary subset of ω_1 , is not stationary.
- For every ordinal γ , S_γ has either a cofinal branch or is special.
- For every ordinal γ , T_γ has either a cofinal branch or is special.

It is important to note that \tilde{W} provides the low-complexity definitions to enable the following absoluteness property between \mathbb{T} -models and generic extensions of \tilde{W} :

Lemma 1.22. *Let $\tilde{W}[G]$ be a generic extension of \tilde{W} which preserves ω_1 . Let $M \in \tilde{W}[G]$ be a transitive \mathbb{T} -model of size \aleph_1 . Then if $r \in M \cap 2^\omega$ and*

$$M \models \forall n \in \omega (n \in r \Leftrightarrow T_{\omega_{\gamma+n}} \text{ has a cofinal branch} \wedge \\ n \notin r \Leftrightarrow T_{\omega_{\gamma+n}} \text{ is special})$$

then

$$\tilde{W}[G] \models \forall n \in \omega (n \in r \Leftrightarrow T_{\omega_{\gamma+n}} \text{ has a cofinal branch} \wedge \\ n \notin r \Leftrightarrow T_{\omega_{\gamma+n}} \text{ is special})$$

Consequently, the reals which are seen to be coded by an \aleph_1 -sized, transitive \mathbb{T} -model into \vec{T} are coded into \vec{T} in the real world.

Proof. We note first that any transitive, \aleph_1 -sized \mathbb{T} -model M will define the sequence \vec{T} up to $M \cap \omega_2$ correctly as long as it has all the information available. Indeed if M is such that

- $M \models \text{ZF}^-$,
- M is transitive,
- $\aleph_1 \in M$, and
- for every pair of ordinals $(2\beta, 2\beta + 1)$ either $R_{2\beta}$ or $R_{2\beta+1}$, where S_β is our definable L -stationary subset of ω_1 , is not stationary.
- For every ordinal γ , S_γ has either a cofinal branch or is special.

Then, using the $\Sigma_1(\omega_1)$ -formula $\Xi(v_0, v_1, \omega_1)$ from Lemma 1.21, M will compute $\vec{T} \upharpoonright (M \cap \omega_2)$ correctly. As “having a cofinal branch”, and “being special” are both $\Sigma_1(\omega_1)$ -properties, the pattern M thinks it sees on its version of the \vec{T} -sequence must be the patterns which we find in $\tilde{W}[G]$, as otherwise ω_1 would have been collapsed. □

As an immediate consequence, \mathbb{T} -models form a well-defined stratification of subsets of $H(\omega_2)$ of $\tilde{W}[G]$, the latter being an ω_1 -preserving generic extension of \tilde{W} .

Lemma 1.23. *Work in a universe $\tilde{W}[G] \supset \tilde{W}$ which satisfies that $\omega_1^{\tilde{W}[G]} = \omega_1^{\tilde{W}}$. Let $M \in \tilde{W}[G]$ be a transitive model of size \aleph_1 which satisfies \mathbb{T} . Then M is uniquely determined by $\omega_2 \cap M$.*

Proof. Every \aleph_1 -sized, transitive \mathbb{T} -model M is, by the definition of \mathbb{T} , completely determined by the reals it contains. As every real in M is coded by an ω -block of Suslin trees $(T_{\omega\gamma+n} : n \in \omega)$, every \mathbb{T} -model is determined by the set of codes which are written on its block of Suslin trees \vec{T} . In $\tilde{W}[G]$, there is exactly one such sequence and it can be read off by transitive, \aleph_1 -sized models \mathbb{T} . Thus, for every $\alpha < \omega_2$, there is at most one transitive \mathbb{T} -model M such that $M \cap \omega_2 = \alpha$. \square

Now we return to defining our iteration with \tilde{W} as the ground model. Two types of forcings are used, one for creating a hierarchy of \mathbb{T} -models tied together with a localization forcing, and the second one for working towards \mathbf{MA}_{ω_1} . We will use finite support and we use a bookkeeping function $F : \omega_2 \rightarrow (\omega_2)^2$ to organize the iteration. The choice of F does not really matter, we will only assume that F is surjective and that for every $(\alpha, \beta) \in (\omega_2)^2$, $F^{-1}(\alpha, \beta)$ is unbounded in ω_2 . The iteration will be defined inductively, suppose we are at a stage $\alpha < \omega_2$ and we have already defined the iteration \mathbb{P}_α up to α and a generic filter G_α for \mathbb{P}_α . We shall define the next forcing \mathbb{Q}_α we want to use next.

- Suppose $\alpha = \beta + 1$ for some ordinal β . Then we let $\dot{\mathbb{Q}}_\alpha$ be a two step iteration $\mathbb{R}_0 * \mathbb{R}_1$ where \mathbb{R}_0 is itself an iteration of length ω_1 , whereas \mathbb{R}_1 will serve as a coding forcing. The forcing \mathbb{R}_0 will produce a new \mathbb{T} -model. We force in ω_1 -many steps a generic extension $\tilde{W}[G_{\alpha+1}]$ of $\tilde{W}[G_\alpha]$ such that there is a new, transitive model M of \mathbb{T} of size \aleph_1 and height $\beta_\alpha < \omega_2$. There are many ways how to achieve this and the definition is not dependent of which one we pick. We let $H(\alpha)$ be a \mathbb{R}_0 -generic filter over $\tilde{W}[G_\alpha]$ and define \mathbb{R}_1 now. Let $X_\alpha \subset \omega_1$ be a set which codes M . First we note that we can rewrite X_α into a new set $Y_\alpha \subset \omega_1$ which has the additional property that its information can be read off already by suitable countable transitive models as we will see later.

The considerations which will yield the desired set Y_α are as follows. Note first that the set of countable $X \prec L_{\omega_2}[M]$ which contain M give rise to a club

$$C := \{\beta < \omega_1 : \exists X \prec L_{\omega_2}[M] \wedge M \in X \wedge \omega_1 \cap X = \beta\}.$$

We let $\{c_\beta : \beta < \omega_1\}$ be the enumeration of C . We define $Y_\alpha \subset \omega_1$ such that the odd entries code the X_α and the even entries $E(Y_\alpha)$ of Y_α satisfy:

1. $E(Y_\alpha) \cap \omega$ codes a well-ordering of type c_0 .

2. $E(Y_\alpha) \cap [\omega, c_0) = \emptyset$.
3. For all β , $E(Y_\alpha) \cap [c_\beta, c_\beta + \omega)$ codes a well-ordering of type $c_{\beta+1}$.
4. For all β , $E(Y_\alpha) \cap [c_\beta + \omega, c_{\beta+1}) = \emptyset$.

In the next step we use almost disjoint forcing $\mathbb{A}_D(Y_\alpha)$ relative to the $<_L$ -least almost disjoint family of reals D to code the set Y_α into one real r_α . The almost disjoint coding forcing shall be our second forcing \mathbb{R}_1 .

- Suppose that α is a limit ordinal, then force towards obtaining a model for \mathbf{MA}_{ω_1} . We pick with the help of the bookkeeping function F the according ccc poset $\mathbb{P} = F(\alpha)$ of size \aleph_1 in a diagonal way. To be more precise, we let (β, γ) be such that $F(\alpha) = (\beta, \gamma)$ and pick the γ -th name (in some fixed well-order of $H(\omega_2)$) of a forcing $\dot{\mathbb{Q}}$ in $\tilde{W}^{\mathbb{P}_\beta}$ which is ccc in $\tilde{W}[G_\alpha]$ and which has the additional property that there is a $\delta < \omega_2$ such that if we write \tilde{W} as $W[G^0][G^1]$, then $\dot{\mathbb{Q}}$ is actually in $W[G_\delta^0][G_\delta^1][G_\alpha]$. If this is the case, then we force with $\dot{\mathbb{Q}}_\alpha := F(\alpha)$, but only if $\dot{\mathbb{Q}}_\alpha$ satisfies that

$\tilde{W}[G_\alpha] \models \dot{\mathbb{Q}}_\alpha \Vdash$ “There are \aleph_2 -many trees in \vec{T} which are still Suslin”.

Otherwise we force with the trivial forcing.

This ends the definition of our iteration. We add as a remark, that at every intermediate stage $\alpha < \omega_2$ of the iteration, by definition, there will still be \aleph_2 -many Suslin trees from our definable sequence \vec{T} which are still Suslin in the intermediate model $L[G_\alpha]$, where G_α as always denotes the generic for the iteration cut at α . Thus in the iteration, we never run out of trees, so the length of it will be ω_2 .

1.5.1 Properties of W_1

After ω_2 -many steps we arrive at a model $\tilde{W}[G_{\omega_2}] =: W_1$ which will satisfy the Σ_3^1 -uniformization property and \mathbf{MA}_{ω_1} as we shall show now. The following is straightforward from the definition of the iteration.

Lemma 1.24. $H(\omega_2)^{W_1} \models \mathbf{T}$. *In particular, every subset of ω_1 is coded by a real and every real is coded into an ω -block of our definable sequence of Suslin trees \vec{T} .*

The upshot of forcing \mathbf{MA}_{ω_1} in the diagonal fashion utilized in the definition of \mathbb{P}_{ω_2} are the two following lemmas

Lemma 1.25. *Let $\alpha < \omega_2$ and let \mathbb{P}_α be the initial segment of \mathbb{P}_{ω_2} . Then*

$$\tilde{W} \models |\mathbb{P}_\alpha| = \aleph_1$$

Proof. The proof is by induction on $\alpha < \omega_2$. For limit stages α , the assertion follows from the fact that we use finite support in our iteration. For successor stages $\alpha + 1$, we note that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ and that by the way we defined \mathbb{P}_{ω_2} , there must be a stage $\beta_\alpha < \omega_2$ such that $\mathbb{P}_\alpha \in W[G_{\beta_\alpha}^0][G_{\beta_\alpha}^1]$. Indeed if α is a limit ordinal, then this follows immediately from our definition of the iteration and if α is successor then this follows from the fact that in the process of producing the next T-model, the only forcings which are used are trees from \vec{T} and almost disjoint coding forcings relative to the canonical almost disjoint family of reals. Likewise there is $\beta_{\alpha+1} < \omega_2$ such that $\dot{\mathbb{Q}}_\alpha$ is in fact in $W[G_{\beta_{\alpha+1}}^0][G_{\beta_{\alpha+1}}^1]$. Now we note that CH is true in $W[G_{\beta_{\alpha+1}}^0][G_{\beta_{\alpha+1}}^1]$, as CH is true in \tilde{W} and both forcings are \aleph_1 -length iterations of ccc forcings over W . This implies that $\dot{\mathbb{Q}}_\alpha$ is essentially an \aleph_1 -sequence of countable antichains in \mathbb{P}_α which is an element of $W[G_{\beta_{\alpha+1}}^0][G_{\beta_{\alpha+1}}^1]$. Hence $|\mathbb{P}_{\alpha+1}| = \aleph_1$, and the lemma is proved. \square

Lemma 1.26. *In W_1 , MA_{ω_1} holds.*

Proof. Assume that $\mathbb{Q} \in W_1 = \tilde{W}[G_{\omega_2}]$ is an \aleph_1 -sized partial order with the ccc and that D is an \aleph_1 -sized family of dense subsets of \mathbb{Q} . Then there is a stage $\beta < \omega_2$, such that, if G_β denotes the generic filter for the intermediate forcing $\mathbb{P}_\beta \subset \mathbb{P}_{\omega_2}$, then \mathbb{Q} and D are already elements of $\tilde{W}[G_\beta]$.

By the (proof of the) last lemma, we know that \mathbb{P}_β is an \aleph_1 -sized forcing as seen from \tilde{W} , and there are ordinals $\alpha_\beta < \omega_2$, such that \mathbb{P}_β is in fact an element of $W[G_{\alpha_\beta}^0][G_{\alpha_\beta}^1]$.

But then, $\mathbb{P}_\beta * \mathbb{Q}$ is an element of $W[G_{\alpha_\beta}^0][G_{\alpha_\beta}^1]$, and $W[G_{\alpha_\beta}^0][G_{\alpha_\beta}^1] \models$ “ $|\mathbb{P}_\beta * \mathbb{Q}| = \aleph_1$ and has the ccc”.

By our choice of the bookkeeping function F , there will be cofinally many stages $\gamma > \beta$ such that the forcing considered by F at stage γ is \mathbb{Q} . In particular there will be a least such stage, denoted again by $\gamma > \beta$, such that $D \in \tilde{W}[G_\gamma]$ and \mathbb{Q} is suggested by F at γ . We claim that at stage γ ,

$$\tilde{W}[G_\gamma] \models \mathbb{Q} \Vdash \text{“There are } \aleph_2\text{-many trees in } \vec{T} \text{ which are still Suslin”}.$$

If the claim is true, then by definition, we must have used \mathbb{Q} at stage γ , and so MA_{ω_1} must be true.

To see why the claim holds, we first let H be \mathbb{Q} -generic and note that if α_γ denotes a stage $< \omega_2$ such that $\mathbb{P}_\gamma \in W[G_{\alpha_\gamma}^0][G_{\alpha_\gamma}^1]$, we can factor $G_{\omega_2}^0$ and $G_{\omega_2}^1$ into $G_{\alpha_\gamma}^i * G_{[\alpha_\gamma, \omega_2]}^i$, and hence

$$W[G_{\omega_2}^0][G_{\omega_2}^1][G_\gamma][H] = W[G_{\alpha_\gamma}^0][G_{\alpha_\gamma}^1][G_\gamma][H][G_{[\alpha_\gamma, \omega_2]}^0][G_{[\alpha_\gamma, \omega_2]}^1].$$

So, in particular, the \aleph_2 -many Cohen reals added by $G_{[\alpha_\gamma, \omega_2]}^0$ will still be Cohen-generic reals when considered over the ground model $W[G_{\alpha_\gamma}^0][G_{\alpha_\gamma}^1][G_\gamma][H]$.
As a consequence

$$W[G_{\alpha_\gamma}^0][G_{\alpha_\gamma}^1][G_\gamma][H][G_{[\alpha_\gamma, \omega_2]}^0][G_{[\alpha_\gamma, \omega_2]}^1] \models \text{“There are } \aleph_2\text{-many trees in } \vec{T} \text{ which are still Suslin.”}$$

hence

$$W[G_{\omega_2}^0][G_{\omega_2}^1][G_\gamma][H] \models \text{“There are } \aleph_2\text{-many trees in } \vec{T} \text{ which are still Suslin.”}$$

Thus the claim is true and we must have used \mathbb{Q} at stage γ . □

We know already that transitive \mathbb{T} -models of size \aleph_1 are uniquely determined by their ordinal height. As we forced with many almost disjoint coding forcings, we obtain a projectively definable set of reals which code \mathbb{T} -models. For a real r we let $(r)_1$ and $(r)_2$ denote its recursive splitting into its even and its odd part respectively.

Lemma 1.27. *In W_1 , for every $\gamma < \omega_2$ there is a \mathbb{T} -model M of ordinal height $> \gamma$, and there is a real r_M which satisfies that $(r_M)_2 = r_{(M \cap \omega_2)}$, where $r_{(M \cap \omega_2)}$ is a real which codes the ordinal $M \cap \omega_2$, and which additionally satisfies:*

$$\begin{aligned} (*) \quad \forall N (|N| = \aleph_0 \wedge N \text{ is transitive and } \omega_1^N = (\omega_1^L)^N \wedge r_M \in N \\ \rightarrow N \models \text{“} L[r_M] \models (r_M)_1 \text{ codes a } \mathbb{T}\text{-model of height } \alpha((r_M)_2), \\ \text{where } \alpha((r_M)_2) \text{ is the ordinal } < \omega_2^N \\ \text{decoded from the real } (r_M)_2\text{.”}) \end{aligned}$$

If N is uncountable and transitive and r_M is as above, then

$$N \models \text{“The model decoded out of } r_M \text{ is a } \mathbb{T}\text{-model.”}$$

Proof. We defined our iteration such that at successor stages, we always produce \mathbb{T} -models and subsequently code them into reals. We let $\gamma < \omega_2$ be arbitrary and consider an $\alpha = \beta + 1 > \gamma$, i.e. a stage past γ where we produce a new \mathbb{T} -model M and a real r which codes the set Y_α , where Y_α is as in the definition of the iteration.

If we let s be a real which codes the ordinal height of M , i.e. a real which codes a set $X_s \subset \omega_1$ which in turn codes a wellorder of ordertype $M \cap \omega_2$,

then we claim that the real x which consists of r on its even, and s on its odd entries is as desired, i.e. $x = r_M$.

To see this, assume that N is countable, transitive, $\omega_1^N = (\omega_1^L)^N$ and $r_M \in N$. Now first note that if N contains r_M , it also must contain $Y_\alpha \cap \omega_1^N$, as the latter is coded into r which in turn is coded into r_M , and N can use its local almost disjoint family of L -reals $D \upharpoonright \omega_1^N = D \upharpoonright (\omega_1^L)^N$ to successfully compute $Y_\alpha \cap \omega_1^N$. This means in particular, that there is a countable $X \prec L_{\omega_2}[M]$, such that the transitive collapse of X , denoted with \bar{X} and N have the same ω_1 . Consequentially, if we consider $(L_{\omega_1}[r_M])^N$, then $(L_{\omega_1}[r_M])^N$ is a countable initial segment of $L[r_M]$ and is equal to the $(L_{\omega_1}[r_M])^{\bar{X}}$ which is formed inside \bar{X} . By elementarity,

$$\bar{X} \models "L_{\omega_1}[r_M] \models (r_M)_1 \text{ codes a } \mathbb{T}\text{-model of height } \alpha((r_M)_2)",$$

and so also

$$N \models "L_{\omega_1}[r_M] \models (r_M)_1 \text{ codes a } \mathbb{T}\text{-model of height } \alpha((r_M)_2)".$$

But this is already the first assertion of the lemma.

To show that for any \aleph_1 -sized, transitive N , $N \models$ "The model decoded out of $(r_M)_1$ is M ", we assume for a contradiction that this is wrong and N is a witness that the assertion is wrong. Then we can pick a countable

$$X \prec N$$

which contains r_M , and its transitive collapse \bar{X} would see that

$$\bar{X} \models "L[r_M] \models (r_M)_1 \text{ does not code a } \mathbb{T}\text{-model}."$$

Now it is sufficient to note that \bar{X} is a model as in the first assertion; indeed it is countable, transitive, contains r_M and its local L computes ω_1 correctly, as N does so. But this is a contradiction. \square

Note that the statement $(*)$ is a $\Pi_2^1(r_M)$ -assertion and as a consequence there is a Π_2^1 -definable set of reals which code, in the sense of $(*)$, \mathbb{T} -models.

The fact that \mathbb{T} -models are stratified by their ordinal height enables the following definition of a well-order of the reals, in fact of $P(\omega_1)$. For $x, y \in 2^\omega$, let M_x be the least \mathbb{T} -model which contains x and likewise define M_y . If $x \in M$, then we can assign an ordinal α_x to x which is the least ordinal such that M thinks that there is an ω -block of Suslin trees from \bar{T} starting at α_x which codes x i.e it is true that

$$n \in x \Leftrightarrow M \models T_{\alpha_x+n} \text{ has a branch.}$$

and

$$n \notin x \Leftrightarrow M \models T_{\alpha_x+n} \text{ is special.}$$

This induces a wellorder of the reals, in fact a wellorder of $P(\omega_1)$. For $x, y \in 2^\omega$, we let

$$x < y \Leftrightarrow \alpha_x < \alpha_y.$$

Lemma 1.28. *In W_1 , the Σ_3^1 -uniformization property holds.*

Proof. Let $\varphi(v_0, v_1) = \exists v_2 \psi(v_0, v_1, v_2)$ be an arbitrary Σ_3^1 -formula with two free variables, let $x \in 2^\omega$ and assume that $W_1 \models \exists y \varphi(x, y)$, thus the x -section of the set $A \subset 2^\omega \times 2^\omega$ defined via $\varphi(v_0, v_1)$ is non-empty.

We collect the set of M where M is a \mathbb{T} -model, for which there is a real r_M coding it, and which contains the reals x, y and sees that $\exists z \psi(x, y, z)$ is true. Note that by Shoenfield absolutness, this implies that $\varphi(x, y)$ really is true. As there is a definable wellorder $<$ of the reals, we pick M such that its $<$ -least code is minimal among all such reals coding such \mathbb{T} -models. With the help of our coding this can be written in a $\Pi_2^1(r_M)$ -way:

$$\forall N (r_M \in N \wedge \omega_1^N = (\omega_1^L)^N \rightarrow N \text{ decodes out of } r_M \text{ a } \mathbb{T}\text{-model } M \text{ such that}$$

$$M \text{ is the } <\text{-least such model which sees that}$$

$$\exists z \psi(x, y, z) \text{ is true.)}$$

The fact that \mathbb{T} -models can internally define a wellorder of its reals, can be used to single out a pair of reals (y, z) which M thinks is the least to satisfy $\psi(x, y, z)$. Then we let y be the first coordinate of this $<$ -least pair just defined and set y to be the value of our uniformizing function f at x . Thus the relation

$$f(x) = y$$

can be defined as follows:

$$(\heartsuit) \quad \exists r \forall N (r \in N \wedge \omega_1^N = (\omega_1^L)^N \rightarrow N \text{ decodes out of } r \text{ a } \mathbb{T}\text{-model } M \text{ such that}$$

$$M \text{ is the } <\text{-least such model which sees that}$$

$$\exists y \exists z \psi(x, y, z) \text{ is true and which thinks that for all}$$

$$\text{pair of reals } (y', z') < (y, z) \varphi(x, y', z') \text{ is wrong.)}$$

Note that this is a Σ_3^1 -formula and we take this to be the definition of f .

What is left is to show that f defined via (\heartsuit) really defines a uniformizing function for the Σ_3^1 -set defined by $\{(x, y) : \varphi(x, y)\}$. We shall show three things, first we show that f is always well-defined; second we show that whenever $f(x) = y$, then $\varphi(x, y)$ is true; and third we show that $\text{dom } f = \{x : \exists y(\varphi(x, y))\}$.

First, we shall show that f is well-defined. Assume for a contradiction that there are $y \neq y'$ such that $f(x) = y$ and $f(x) = y'$ defined via (\heartsuit) is true and let r_y and $r_{y'}$ be the reals which witness the truth of $f(x) = y$ and $f(x) = y'$ in (\heartsuit) . Now, if $H(\theta)$ is sufficiently big, then by lemma 1.27 it will decode out of r_y and $r_{y'}$ two \mathbb{T} -models M_y and $M_{y'}$ which are both $<$ -least for thinking that $\exists y \exists z (\psi(x, y, z))$ is true in M_y and $M_{y'}$. So $M_y = M_{y'}$ and hence $y = y'$. Thus $f(x)$ is welldefined.

Second we assume that $f(x) = y$ holds, and let r be the real which witnesses that the property (\heartsuit) is true. Then, by lemma 1.27, if N is some sufficiently big $H(\theta)$, it is clear that r is in fact a code for a true \mathbb{T} -model M , which in turn must think that $\varphi(x, y)$ is true. But then, by Shoenfield absoluteness, $\varphi(x, y)$ is true so $f(x) = y$ implies that $\varphi(x, y)$ is true.

Third, if x is such that there is a y such that $\varphi(x, y)$ is true, then there is a \mathbb{T} -model M which sees that as well. Surely there will be a bigger \mathbb{T} -model $M' \supset M$ which contains r_M , where the latter is a real which codes M . As M' is coded by some real $r_{M'}$ we have that $r_{M'}$ must satisfy (\heartsuit) , hence $x \in \text{dom} f$. This shows that if $\exists y (\varphi(x, y))$, then $x \in \text{dom} f$. The other inclusion is already shown in the second claim, so the third claim is shown. \square

The above generalizes to the boldface case immediately.

Corollary 1.29. *In W_1 for every real r , every $\Sigma_3^1(r)$ -relation in the plane can be uniformized by a $\Sigma_3^1(r)$ -definable function.*

2 BPFA and the Σ_3^1 -uniformization property.

The results of the last section can be strengthened if we assume BPFA. We aim to prove the following result.

Theorem 2.1. *Assume BPFA and $\omega_1 = \omega_1^L$, then the Σ_3^1 -uniformization property holds.*

Note that the above is not a consistency result. Its proof makes heavy use of a coding method invented by A. Caicedo and B. Velickovic (see [4]) as well as an important observation by A. Caicedo and S. D. Friedman that one can easily combine the Coding method with David's trick (see [3]). As it is somewhat difficult to describe their technique in a short way without leaving out too many important features out, we opt to introduce their coding method briefly.

Definition 2.2. A \vec{C} -sequence, or a ladder system, is a sequence $(C_\alpha : \alpha \in \omega_1, \alpha \text{ a limit ordinal})$, such that for every α , $C_\alpha \subset \alpha$ is cofinal and the order type of C_α is ω .

As we always work with L as our ground model, there is a canonical ladder system $\vec{C} \in L$ which can be defined using L 's Δ_2^1 -definable well-order of the reals. From now on whenever we write \vec{C} , we have this canonical ladder system in mind.

For three subsets $x, y, z \subset \omega$ we can consider the oscillation function. First, turn the set x into an equivalence relation \sim_x , defined on the set $\omega - x$ as follows: for natural numbers in the complement of x satisfying $n \leq m$, let $n \sim_x m$ if and only if $[n, m] \cap x = \emptyset$. This enables us to define:

Definition 2.3. For a triple of subset of the natural numbers (x, y, z) list the intervals $(I_n : n \in k \leq \omega)$ of equivalence classes of \sim_x which have nonempty intersection with both y and z . Then, the oscillation map $o(x, y, z) : k \rightarrow 2$ is defined to be the function satisfying

$$o(x, y, z)(n) = \begin{cases} 0 & \text{if } \min(I_n \cap y) \leq \min(I_n \cap z) \\ 1 & \text{else.} \end{cases}$$

Next, we want to define how suitable countable subsets of ordinals can be used to code reals. For that suppose that $\omega_1 < \beta < \gamma < \delta$ are fixed limit ordinals, and that $N \subset M$ are countable subsets of δ . Assume further that $\{\omega_1, \beta, \gamma\} \subset N$ and that for every $\eta \in \{\omega_1, \beta, \gamma\}$, $M \cap \eta$ is a limit ordinal and $N \cap \eta < M \cap \eta$. We can use (N, M) to code a finite binary string. Namely, let \bar{M} denote the transitive collapse of M , let $\pi : M \rightarrow \bar{M}$ be the collapsing map and let $\alpha_M := \pi(\omega_1)$, $\beta_M := \pi(\beta)$, $\gamma_M := \pi(\gamma)$ $\delta_M := \bar{M}$. These are all countable limit ordinals. Furthermore set $\alpha_N := \sup(\pi''(\omega_1 \cap N))$ and let the height $n(N, M)$ of α_N in α_M be the natural number defined by

$$n(N, M) := \text{card}(\alpha_N \cap C_{\alpha_M}),$$

where C_{α_M} is an element of our previously fixed ladder system. As $n(N, M)$ will appear quite often in the following we write shortly n for $n(N, M)$. Note that as the order type of each C_α is ω , and as $N \cap \omega_1$ is bounded below $M \cap \omega_1$, n is indeed a natural number. Now, we can assign to the pair (N, M) a triple (x, y, z) of finite subsets of natural numbers as follows:

$$x := \{\text{card}(\pi(\xi) \cap C_{\beta_M}) : \xi \in \beta \cap N\}.$$

Note that x again is finite as $\pi''(\beta \cap N)$ is bounded in the cofinal in β_M -set C_{β_M} , which has ordertype ω . Similarly we define

$$y := \{\text{card}(\pi(\xi) \cap C_{\gamma_M}) : \xi \in \gamma \cap N\}$$

and

$$z := \{\text{card}(\pi(\xi) \cap C_{\delta_M}) : \xi \in \delta \cap N\}.$$

Again, it is easily seen that these sets are finite subsets of the natural numbers. We can look at the oscillation $o(x \setminus n, y \setminus n, z \setminus n)$ and if the oscillation function at these points has a domain bigger or equal to n then we write

$$s_{\beta, \gamma, \delta}(N, M) := \begin{cases} o(x \setminus n, y \setminus n, z \setminus n) \upharpoonright n & \text{if defined} \\ * & \text{else.} \end{cases}$$

We let $s_{\beta, \gamma, \delta}(N, M) \upharpoonright l = *$ when $l \geq n$. Finally we are able to define what it means for a triple of ordinals (β, γ, δ) to code a real r .

Definition 2.4. *For a triple of limit ordinals (β, γ, δ) , we say that it codes a real $r \in 2^\omega$ if there is a continuous increasing sequence $(N_\xi : \xi < \omega_1)$ of countable sets of ordinals, also called a reflecting sequence, whose union is δ and which satisfies that whenever $\xi < \omega_1$ is a limit ordinal then there is a $\nu < \xi$ such that*

$$r = \bigcup_{\nu < \eta < \xi} s_{\beta, \gamma, \delta}(N_\eta, N_\xi).$$

The technicalities in the definitions are justified by the fact that BPFA suffices to introduce witnesses to the codings.

Theorem 2.5 (Caicedo-Velickovic). *Assume that BPFA holds, then:*

- (†) *Given ordinals $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 , then there is a reflecting, i.e., increasing and continuous sequence $(N_\xi : \xi < \omega_1)$ such that $N_\xi \in [\delta]^\omega$ whose union is δ such that for every limit $\xi < \omega_1$ and every $n \in \omega$ there is $\nu < \xi$ and $s_\xi^n \in 2^n$ such that*

$$s_{\beta, \gamma, \delta}(N_\eta, N_\xi) \upharpoonright n = s_\xi^n$$

holds for every η in the interval (ν, ξ) . We say then that the triple (β, γ, δ) is stabilized.

- (‡) *Further if we fix a real r there is a triple of ordinals $(\beta_r, \gamma_r, \delta_r)$ of size and cofinality \aleph_1 and a reflecting sequence $(P_\xi : \xi < \omega_1)$, $P_\xi \in [\delta_r]^\omega$ such that $\bigcup_{\xi < \omega_1} P_\xi = \delta_r$ and such that for every limit $\xi < \omega_1$ there is a $\nu < \xi$ such that*

$$\bigcup_{\nu < \eta < \xi} s_{\beta_r, \gamma_r, \delta_r}(P_\eta, P_\xi) = r.$$

We say then that the real r is coded by the triple $(\beta_r, \gamma_r, \delta_r)$.

The coding induces a hierarchy on $H(\omega_2)$ whose initial segments are $\Sigma_1(\omega_1)$ -definable. As we work over L , we can easily define ladder system \vec{C} via inductively picking always the $<_L$ -least real coding a cofinal set. From now on we exclusively work with this ladder system \vec{C} . Recall that we fixed a similarly definable, almost disjoint family of reals D .

Definition 2.6. Let $\mathbb{T}_{\vec{C}}$ denote the following list of axioms:

1. $\forall x(|x| \leq \aleph_1)$,
2. ZF^- ,
3. Every subset of ω_1 is coded by a real, relative to the almost disjoint family D .
4. Every triple of limit ordinals is stabilized in the sense of \dagger using \vec{C} .
5. Every real is determined by a triple of ordinals in the sense of \ddagger using \vec{C} .

A highly useful feature of models of $\mathbb{T}_{\vec{C}}$ is that they are uniquely determined by their height, consequentially the uncountable $\mathbb{T}_{\vec{C}}$ -models form a hierarchy below $H(\omega_2)$.

Theorem 2.7. Let \vec{C} be a ladder system in M , assumed to be a transitive model of $\mathbb{T}_{\vec{C}}$. Then, M is the unique model of $\mathbb{T}_{\vec{C}}$ of height $M \cap \text{Ord}$.

Proof. Assume that M and M' are transitive, $M \cap \text{Ord} = M' \cap \text{Ord}$, $\vec{C} \in M \cap M'$, which implies that M and M' have the right ω_1 , and both M and M' are $\mathbb{T}_{\vec{C}}$ -models. We work towards a contradiction, so assume that $X \in M$ yet $X \notin M'$. As every set in M has size at most \aleph_1 we can assume that $X \subset \omega_1$, hence there is a real $r_X \in M$ which codes X with the help of the almost disjoint family $F_{\vec{C}}$. Now r_X is itself coded by a triple of ordinals $(\beta, \gamma, \delta) \in M$, thus there is a reflecting sequence $(N_\xi : \xi < \omega_1) \in M$ witnessing that r_X is determined by (β, γ, δ) . As $M \cap \text{Ord} = M' \cap \text{Ord}$, (β, γ, δ) is in M' as well, and there is a reflecting sequence $(P_\xi : \xi < \omega_1) \in M'$ which witnesses that (β, γ, δ) is stabilized in M' . The set $C := \{\xi < \omega_1 : P_\xi = N_\xi\}$ is a club on ω_1 , hence if η is a limit point of C , the reflecting sequence $(P_\xi : \xi < \omega_1) \in M'$ will stabilize at η and compute r_X , hence X is an element of M' which is a contradiction. \square

It is a fact that under BPFA and $\omega_1 = \omega_1^L$, every subset of ω_1 is coded by a real relative to the canonical almost disjoint family of L -reals D . In particular we will have reals r_M which code the $\mathbb{T}_{\vec{C}}$ -model M in exactly the same way as in Lemma 1.27. We immediately obtain:

Lemma 2.8. *Assume that BPFA is true and $\aleph_1 = \aleph_1^L$. Then for every $\mathbb{T}_{\vec{C}}$ -model M , there is a real r_M such that*

$$(*) \quad \forall N(|N| = \aleph_0 \wedge N \text{ is transitive and } \omega_1^N = (\omega_1^L)^N \wedge r_M \in N \\ \rightarrow N \models \text{“}L[r_M] \models (r_M) \text{ codes a } \mathbb{T}_{\vec{C}}\text{-model”}$$

If N is uncountable and transitive, and r_M is as above, then

$$N \models \text{“The model decoded out of } r_M \text{ is a } \mathbb{T}_{\vec{C}}\text{-model.”}$$

We aim now to argue for the Σ_3^1 -uniformization property. Let $\varphi(v_0, v_1)$ be an arbitrary Σ_3^1 -formula and assume that $x \in 2^\omega$ is such that there is a y such that $\varphi(x, y)$ holds. We define the uniformizing function f as in the last section:

$$f(x) = y \Leftrightarrow \exists M(M \text{ transitive} \wedge \omega_1 \in M \wedge M \models \mathbb{T}_{\vec{C}} \wedge \\ M \models \text{“}\varphi(x, y) \wedge y \text{ is } < \text{-minimal”} \wedge \\ M \models \nexists N(N \models \mathbb{T}_{\vec{C}} \wedge \varphi(x, y)).$$

By Shoenfield absoluteness, it is clear that f does uniformize every Σ_3^1 -subset of the plane. The above Lemma allows us to localize the above definition to obtain a Σ_3^1 -definable uniformizing function.

Lemma 2.9. *Assume BPFA is true and $\aleph_1 = \aleph_1^L$. Let \vec{C} be the canonical ladder system on ω_1^L . Then the Σ_3^1 -uniformization property is true. Any Σ_3^1 -formula $\varphi(v_0, v_1)$ can be uniformized by the function f_φ which is defined by the following Σ_3^1 -formula*

$$f_\varphi(x) = y \Leftrightarrow \exists r \forall N(N \text{ countable and transitive} \wedge \omega_1^N = (\omega_1^L)^N \wedge r \in N \rightarrow \\ N \models r \text{ codes the least } \mathbb{T}_{\vec{C}}\text{-model } M \text{ which sees that} \\ \exists y' \varphi(x, y') \text{ is true and } y \text{ is the least such witness in } M).$$

Its proof is entirely analogous to the proof of Lemma 1.28, so we skip it.

Going from lightface to boldface causes no difficulties at all and we immediately obtain the next result.

Corollary 2.10. *Assume BPFA and that ω_1 is accessible to some real r . Then every $\Sigma_3^1(x)$ relation in the plane can be uniformized by a $\Sigma_3^1(r, x)$ -function.*

3 Some questions

We end with a couple of open questions whose answers would require different methods than the one introduced here.

Question 1. *Is the Σ_n^1 -uniformization property for $n > 3$ and MA_{ω_1} or BPFA consistent?*

Question 2. *Is the Σ_3^1 -uniformization property and a large continuum ($> \aleph_2$) consistent?*

Finally, it is natural to ask whether it is possible to have a global behaviour of the uniformization property in the presence of forcing axioms.

Question 3. *Is there a model of MA_{ω_1} for which the uniformization property simultaneously holds true for Σ_n^1 , $n \geq 3$?*

References

- [1] U. Abraham *Proper Forcing*, Handbook of Set Theory Vol.1. Springer
- [2] J. W. Addison *Some Consequences of the Axiom of Constructibility*. Fundamenta Mathematicae 46, pp. 337-357, 1959.
- [3] A. Caicedo and S.D. Friedman *BPFA and Projective Wellorderings of the Reals*. Journal of Symbolic Logic 76, pp. 1126-1136, (4) 2011.
- [4] A. Caicedo and B. Velickovic *The Bounded Proper Forcing Axiom and Well-Orderings of the Reals*. Mathematical Research Letters 13 (2006), no. 3, 393-408.
- [5] V. Fischer and S.D. Friedman *Cardinal characteristics and projective wellorders*. Annals of Pure and Applied Logic 161, pp. 916-922, 2010.
- [6] M. Goldstern *A Taste of Proper Forcing*. Di Prisco, Carlos Augusto (ed.) et al., Set theory: techniques and applications. Proceedings of the conferences, Curaçao, Netherlands Antilles, June 26-30, 1995 and Barcelona, Spain, June 10-14, 1996. Dordrecht: Kluwer Academic Publishers. 71-82 (1998).
- [7] S. Höffelner *NS_{ω_1} saturated and Δ_1 -definable*. Journal of Symbolic Logic, 86(1), pp. 25-59, 2021.
- [8] S. Höffelner *Forcing the Σ_3^1 -Separation Property*. Journal of Mathematical Logic, accepted for publication, 2022.

- [9] T. Jech *Set Theory*.
- [10] A. Kechris *Classical Descriptive Set Theory*. Springer 1995.
- [11] T. Miyamoto ω_1 -Suslin trees under countable support iterations. *Fundamenta Mathematicae*, vol. 143 (1993), pp. 257–261.
- [12] J. Steel PFA implies $AD^{L(\mathbb{R})}$. *Journal of Symbolic Logic* Vol. 70, No. 4, pp. 1255-1296, 2005.